MATH 1010E Week 3 (Martin Li)

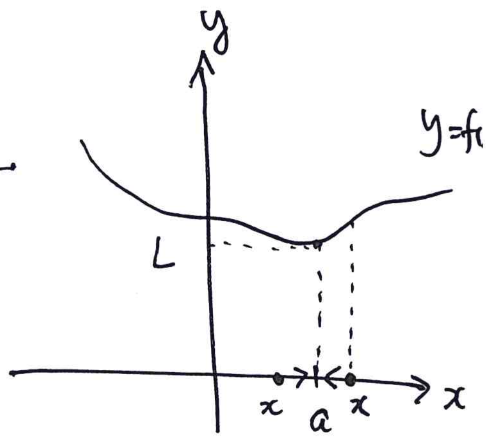
Last time ... " $\lim_{x \to a} f(x) = L$ "

Rule 1 & 2 : Sub $x = a$ into $f(x)$ after simplification.

Left-hand & Right-hand limits

Define: $\lim_{x \to a^+} f(x) := \lim_{\substack{x \to a \\ x > a}} f(x)$ right-hand

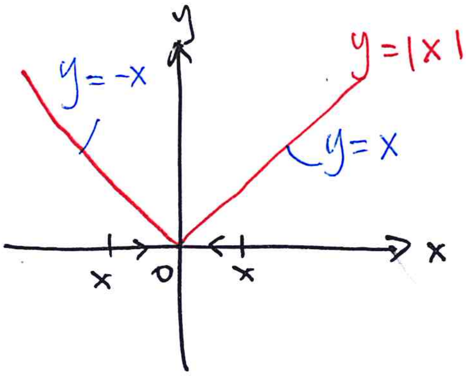
$\lim_{x \to a^-} f(x) := \lim_{\substack{x \to a \\ x < a}} f(x)$ left-hand



Fact: $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$

(ie. limits exist & equal)

Example: $f(x) = |x|$. $\lim_{x \to 0} |x| = ?$



$$|x| := \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x < 0 \end{cases}$$

$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$

$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} -x = -0 = 0$

$\overset{\text{Fact}}{\Longrightarrow} \lim_{x \to 0} |x| = 0$. #

Limits of Piecewise-defined functions

E.g. 1: Define $f(x) = \begin{cases} x+1 & \text{when } x>0 \\ x^2+1 & \text{when } x<0 \end{cases}$

Calculate $\lim_{x\to 0} f(x)$.

Sol:

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+1) = 0+1 = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2+1) = 0^2+1 = 1$$

$1 = 1$ $\overset{\text{Fact}}{\Longrightarrow}$ $\lim_{x\to 0} f(x) = 1$

E.g. 2: Define $f(x) = \begin{cases} x+2 & \text{when } x>0 \\ x^2+1 & \text{when } x<0 \end{cases}$

Calculate $\lim_{x\to 0} f(x)$.

Sol:

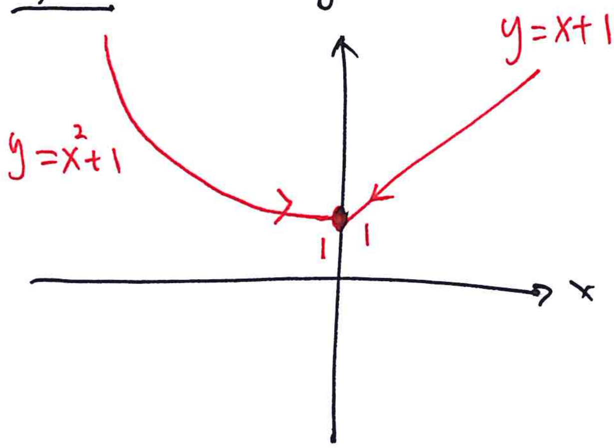
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x+2) = 2$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2+1) = 1$$

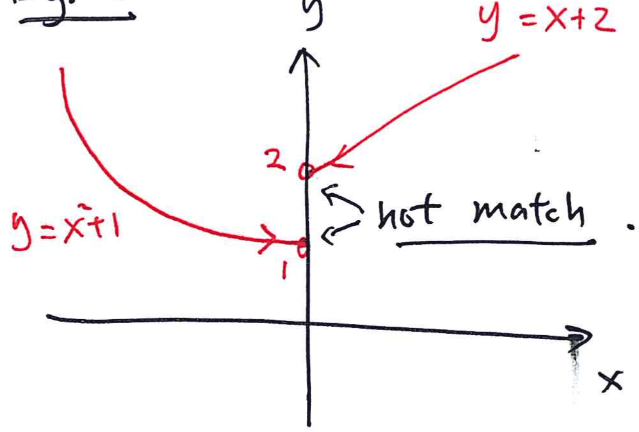
$2 \neq 1$ $\overset{\text{Fact}}{\Longrightarrow}$ $\lim_{x\to 0} f(x)$ does <u>not</u> exist.

Q: What goes wrong?

Eg. 1:



E.g. 2:



Rational Functions

$$f(x) := \frac{P_n(x)}{q_m(x)} \qquad \text{where } P_n(x), q_m(x) \text{ are polynomials.}$$

$$\deg P_n(x) = n$$

$$\deg q_m(x) = m$$

E.g. (1) $$\lim_{x\to\infty} \frac{x^2+2}{x+1} = \lim_{x\to\infty} \frac{x + 2/x}{1 + 1/x} = \infty.$$

(2) $$\lim_{x\to\infty} \frac{x+1}{x^2+2} = \lim_{x\to\infty} \frac{1/x + 1/x^2}{1 + 2/x^2} = 0.$$

(3) $$\lim_{x\to\infty} \frac{2x^2+3}{x^2+2} = \lim_{x\to\infty} \frac{2 + 3/x^2}{1 + 2/x^2} = 2$$

Summarize:

$$\lim_{x\to\infty} \frac{P_n(x)}{q_m(x)} = \begin{cases} \infty & \text{if } n > m \\ 0 & \text{if } m > n \\ \dfrac{a_n}{b_n} & \text{if } m = n \end{cases}$$

where $P_n(x) = a_n x^n + \cdots$

$q_n(x) = b_n x^n + \cdots$

More Properties of Limits $(+, -, \cdot, \div)$

(1) $\displaystyle \lim_{x \to a} \left(f(x) \pm g(x) \right) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

(2) $\displaystyle \lim_{x \to a} \left(f(x)\, g(x) \right) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

(3) $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ when $\displaystyle \lim_{x \to a} g(x) \neq 0$.

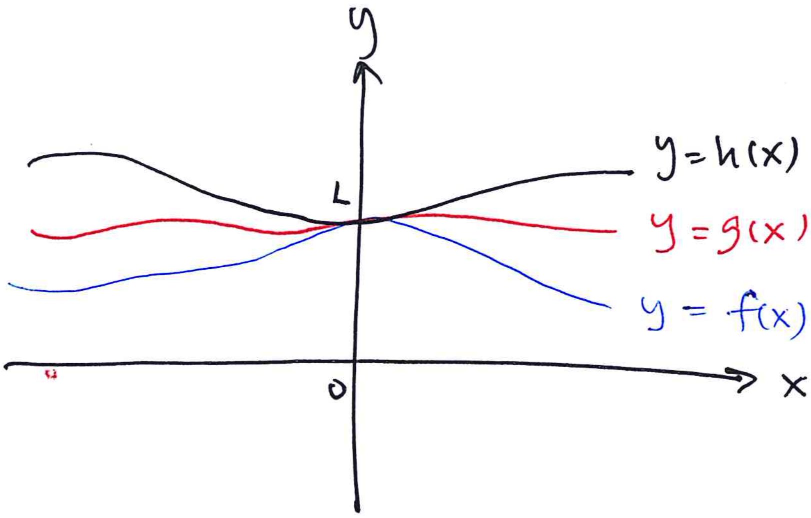
<u>Remark:</u> Requires all the limits on the R.H.S. are defined.

Comparison Theorem

If $f(x) \leq g(x) \quad \forall x \in \mathbb{R}$, then $\displaystyle \lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$

<u>Corollary</u> (Sandwich Thm.)

$$\left.\begin{array}{l} \text{If } \overset{\nearrow L}{f(x)} \leq g(x) \leq \overset{\nearrow L}{h(x)} \quad \forall x \in D \\ \text{and } \displaystyle\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x) \end{array}\right\} \Longrightarrow \lim_{x \to a} g(x) = L$$



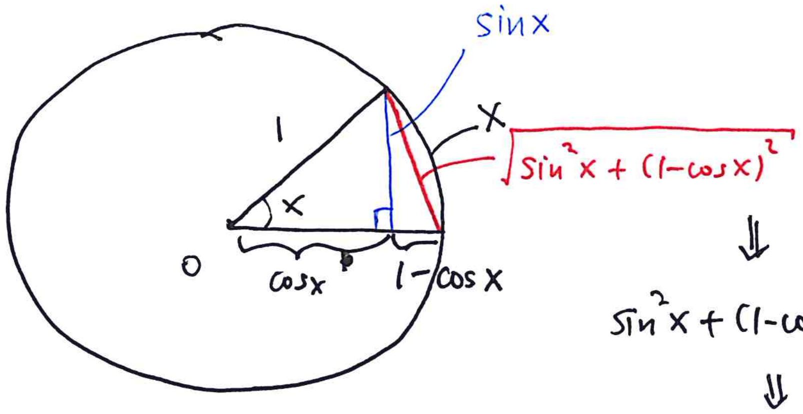
Example: $\lim_{x \to 0} (\sin x) = 0$ by sandwich thm.

Solution: **Claim:** $-|x| \leq \sin x \leq |x| \quad \forall x \in \mathbb{R}$.

Assume claim; then

$$\lim_{x \to 0} -|x| = 0 = \lim_{x \to 0} |x| \overset{\text{Sandwich}}{\Longrightarrow} \lim_{x \to 0} \sin x = 0.$$

Pf of Claim geometrically:



$$\sqrt{\sin^2 x + (1-\cos x)^2} \leq x$$

$$\Downarrow$$

$$\sin^2 x + (1-\cos x)^2 \leq x^2$$

$$\Downarrow$$

$$|\sin x| \leq |x|.$$

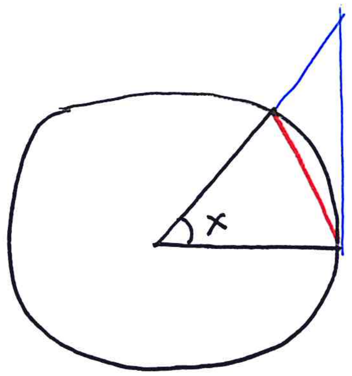
$\blacksquare$

Example: $\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$

Claim: $\cos x < \frac{\sin x}{x} < 1$

$$\lim_{x \to 0} \cos x = 1 = \lim_{x \to 0} 1 \overset{\text{Sandwich}}{\Longrightarrow} \lim_{x \to 0} \frac{\sin x}{x} = 1$$

Ex:



area($\triangle$) < area($\triangle$) < area($\triangle$

Q: finish this

Another way: Recall

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{\sin x}{x} = 1 - \underbrace{\frac{x^2}{3!}}_{\xrightarrow{x\to 0}\ 0} + \underbrace{\frac{x^4}{5!}}_{\xrightarrow{x\to 0}\ 0} \underbrace{- \cdots}_{\xrightarrow{x\to 0}\ 0}$$

On the other hand,

$$\lim_{x\to 0} \frac{\cos x - 1}{x} \overset{?}{=} 0 \quad ※$$

Recall: $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\frac{\cos x - 1}{x} = \left(-\frac{x}{2!} + \frac{x^3}{4!} - \cdots\right) \xrightarrow[x\to 0]{} 0$$

Q: Calculate the limit using double angle formula.

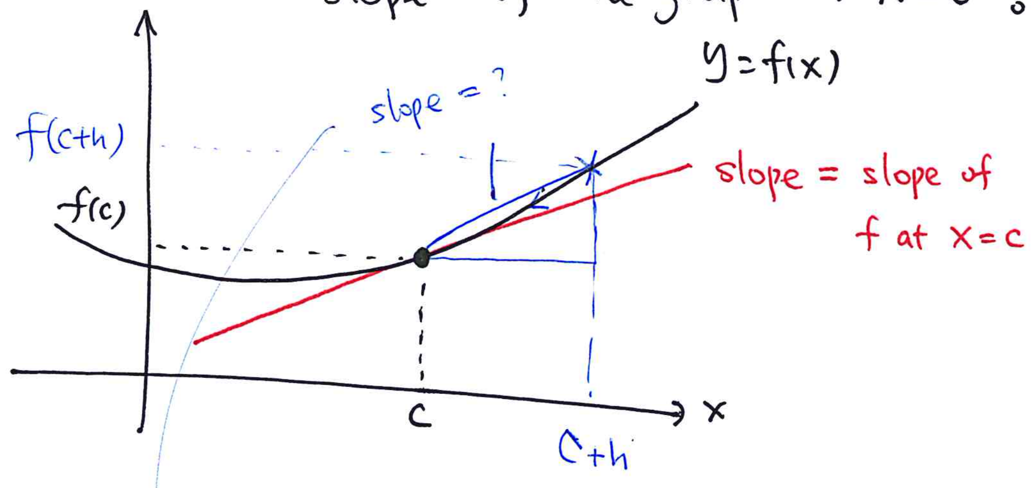
and $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

E.g. $\lim_{x\to 0} \frac{\sin 2x}{5x} = \lim_{x\to 0} \frac{2}{5}\cdot\left(\frac{\sin 2x}{2x}\right) = \frac{2}{5}\cdot 1 = \frac{2}{5}$.

Ex: $\lim_{x\to \infty} \frac{\sin x}{x} = ?$

Limit and Difference Quotient

Q: Given a function $y = f(x)$, how to calculate the "slope" of the graph at $x = c$?



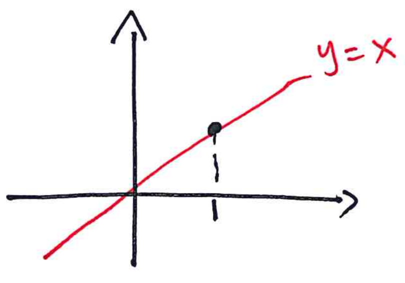
$$\text{Slope} = \frac{f(c+h) - f(c)}{h} =: \left.\frac{\Delta f}{\Delta x}\right|_{x=c} \quad \text{Difference Quotient}$$

$$\underbrace{f'(c) = \left.\frac{df}{dx}\right|_{x=c} := \lim_{h\to 0} \left.\frac{\Delta f}{\Delta x}\right|_{x=c}(h) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h}}_{\text{derivative of } f \text{ at } x=c}.$$

Easy Examples

(1) $f(x) = x$ at $x = 1$

Calculate $f'(1)$.



Sol: Diff. Quotient $\left.\frac{\Delta f}{\Delta x}\right|_{x=1} := \frac{f(1+h) - f(1)}{h} = \frac{(\cancel{1}+h) - \cancel{1}}{h} = 1$

$$f'(1) = \lim_{h\to 0} \left.\frac{\Delta f}{\Delta x}\right|_{x=1} = \lim_{h\to 0} 1 = 1 \quad ※$$

In fact, $f'(c) = 1$ for any $c \in \mathbb{R}$.

(2) $f(x) = x^2$ at $x = 1$.

Calculate $f'(1)$.

Sol: Diff Quo.: $$\left.\frac{\Delta f}{\Delta x}\right|_{x=1} := \frac{f(1+h) - f(1)}{h}$$

$$= \frac{(1+h)^2 - 1^2}{h}$$

$$= \frac{(\cancel{1} + 2h + h^2) - \cancel{1}}{h}$$

$$= 2 + h$$

$$f'(1) = \lim_{h \to 0} \left.\frac{\Delta f}{\Delta x}\right|_{x=1} = \lim_{h \to 0} (2+h) = 2 \quad \divideontimes.$$

Ex: $f(x) = x^n$, $x = 1 \Rightarrow f'(1) = n$.

More complicated examples

- Derivative as a new function:

$$f(x) \rightsquigarrow \underset{\text{at } x=c}{f'(c)} \overset{\text{varie } c}{\rightsquigarrow} f'(x)$$

Eg $f(x) = x^n \rightsquigarrow f'(x) = n x^{n-1}$.

Eg. 1 : $f(x) = \sqrt{x+2}$. Calculate $f'(x)$.

Sol : Diff. Quotient

$$\frac{\Delta f}{\Delta x} := \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\sqrt{(x+h)+2} - \sqrt{x+2}}{h}$$

(rationalize) $$= \frac{(\sqrt{x+h+2} - \sqrt{x+2})(\sqrt{x+h+2} + \sqrt{x+2})}{h(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \frac{(\cancel{x} \cancel{+h} \cancel{+2}) - (\cancel{x} \cancel{+2})}{\cancel{h}(\sqrt{x+h+2} + \sqrt{x+2})}$$

$$= \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}}$$

$$f'(x) = \lim_{h \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} = \frac{1}{2\sqrt{x+2}}$$

#.

E.g. 2: $f(x) = \sin x$ Calculate $f'(x)$?.

Sol: Diff. Quotient ?

$$\frac{\Delta f}{\Delta x} := \frac{f(x+h) - f(x)}{h}$$

$$= \frac{\sin(x+h) - \sin x}{h}$$

$$= \frac{(\sin x \cosh + \cos x \sinh) - \sin x}{h}$$

$$= \sin x \left(\frac{\cosh - 1}{h}\right) + \cos x \left(\frac{\sinh}{h}\right)$$

Take limit.

$$f'(x) := \lim_{h \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \left[\sin x \left(\frac{\cosh - 1}{h}\right) + \cos x \left(\frac{\sinh}{h}\right)\right]$$

$$= \sin x \Big(\underbrace{\lim_{h \to 0} \frac{\cosh - 1}{h}}_{= 0}\Big) + \cos x \Big(\underbrace{\lim_{h \to 0} \frac{\sinh}{h}}_{= 1}\Big)$$

$$= \cos x$$ ※.

$$\boxed{(\sin x)' = \cos x}$$

Ex: $(\cos x)' = -\sin x$

Last time ...

- introduced $f'(c)$, $\left.\frac{df}{dx}\right|_{x=c}$, $\frac{\Delta f}{\Delta x}$
- $\frac{d}{dx}(x^n) = n x^{n-1}$; $\frac{d}{dx}(\sin x) = \cos x$; $\frac{d}{dx}(\cos x) = -\sin x$.

Differentiability

Always: Let $f : (a,b) \longrightarrow \mathbb{R}$, fix $c \in (a,b)$.

if $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$ exists,

then f is differentiable at $x = c$.

and the limit is called the derivative of f at c.

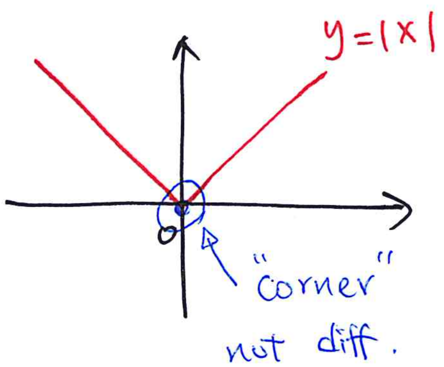
denote: $f'(c)$ or $\left.\frac{df}{dx}\right|_{x=c}$

Remark: $\lim_{x \to c} f(x)$ does not require f defined at $x = c$

but f has to be defined at $x = c$ to talk about derivative at c.

Note: Not all functions are differentiable.

Eg. $f(x) = |x|$. , at $x = 0$



Claim: f is NOT diff. at 0.

$$\frac{\Delta f}{\Delta x} = \frac{f(0+h) - f(0)}{h} = \frac{|h| - 0}{h} = \frac{|h|}{h}$$

$\lim_{h \to 0} \frac{|h|}{h}$ does not exist.

L.H.: $\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$

$\neq$

R.H: $\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = +1$

Ex: Show that the function below is not diff. at 0.

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

(Hint: Sandwich Thm.). Q: Understand why it fails if you at the graph.

~~Continuous~~ **Continuity**: $f: (a,b) \to \mathbb{R}$, $c \in (a,b)$

if $\lim_{x \to c} f(x) = f(c)$ then f is continuous at c

exists equal

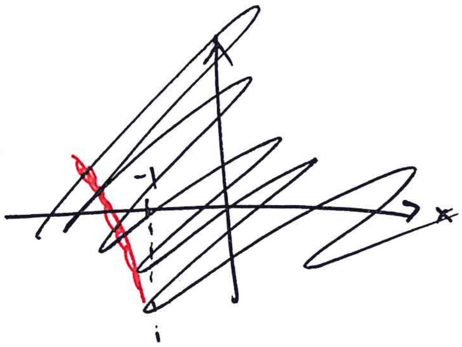
Remark: f has to be defined at c to talk about continuity.

E.g. of cont. functions

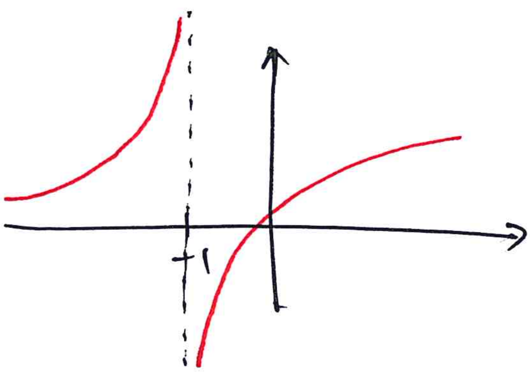
- polynomials: $\lim_{x \to 1} (x^2 + 2x - 1) = 1^2 + 2 \cdot 1 - 1 = 2$
- $\cos x$, $\sin x$, e^x

More examples

(1) $$f(x) = \begin{cases} \frac{x-1}{x+1} & \text{when } x \neq -1 \\ 0 & \text{when } x = -1 \end{cases}$$



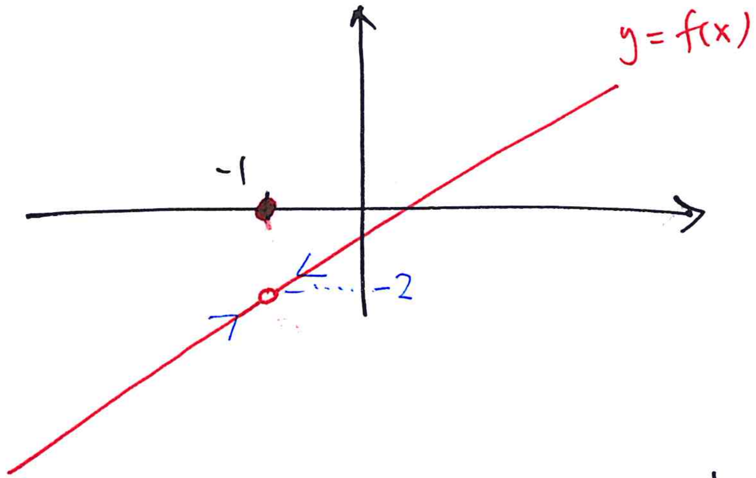
not cts at $x = -1$. $\lim_{x \to -1} f(x)$ does not exists. $(= \pm\infty)$



(2) $$f(x) = \begin{cases} \dfrac{x^2-1}{x+1} & \text{if } x \neq -1 \\ 0 & \text{if } x = -1. \end{cases}$$

<u>Q</u>: Is f cts. at $x=-1$? <u>NO</u>!

$$\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{x^2-1}{x+1} = \lim_{x\to -1} \frac{\cancel{(x+1)}(x-1)}{\cancel{x+1}} = -2 \neq 0 = f(-1)$$



<u>Note</u>: If we define $f(-1) = -2$ then f is cts at -1.

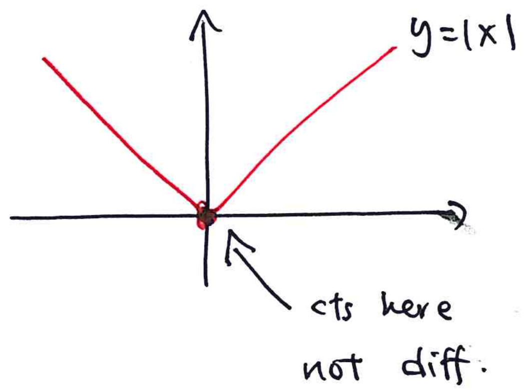
<u>Ex</u>: Show that $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is cts at $x=0$.

<u>Differentiability vs Continuity</u>

$f: (a,b) \longrightarrow \mathbb{R}$, $c \in (a,b)$ fixed.

<u>Thm</u>: f is diff. at $c \implies f$ is cts at c.

<u>Remark</u>: $\Leftarrow$ is FALSE: e.g. $f(x) = |x|$.



Pf: Suppose f is differentiable at $x=c$.

$$\Rightarrow \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \text{ exists and } = f'(c)$$

To show continuity at $x=c$,

Goal: $\boxed{\lim_{x\to c} f(x) = f(c)}$

i.e. $\lim_{h\to 0} f(c+h) = f(c)$

$$\lim_{h\to 0} f(c+h) = \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h}\cdot h + f(c)\right]$$

$$= \underbrace{\lim_{h\to 0}\left(\frac{f(c+h)-f(c)}{h}\right)}_{=f'(c)}\cdot \underbrace{\lim_{h\to 0} h}_{=0} + \underbrace{\lim_{h\to 0} f(c)}_{=f(c)}$$

$$= f(c).$$

#

Differentiation Rules I

(1) $[f(x)\pm g(x)]' = f'(x)\pm g'(x)$

(2) $[k f(x)]' = k f'(x)$

Last time: $\frac{d}{dx}(x^n) = n x^{n-1}$ for $n\in\mathbb{N}$ (Q: $n\in\mathbb{R}$).

$$\Rightarrow \frac{d}{dx}\left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right)$$

$$= a_n n x^{n-1} + a_{n-1}\cdot(n-1)x^{n-2} + \cdots + a_1$$

polynomials.

$$\frac{d}{dx}(\sin x) = \cos x .$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\frac{d}{dx}\left(\sin x\right) = 1 - \frac{\overset{1}{\cancel{3}}\cdot x^2}{\underset{2}{\cancel{3}!}} + \frac{\overset{1}{\cancel{5}}\cdot x^4}{\underset{4}{\cancel{5}!}} - \cdots = \cos x .$$

$\underline{Q}$: $\frac{d}{dx}(e^x) = e^x$.